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Information entropy of Gegenbauer polynomials and Gaussian quadrature

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Abstract

In a recent paper (Buyarov V S, López-Artés P, Martínez-Finkelshtein A and Van Assche W 2000 *J. Phys. A: Math. Gen.* **33** 6549–60), an efficient method was provided for evaluating in closed form the information entropy of the Gegenbauer polynomials $C_n^{(\lambda)}(x)$ in the case when $\lambda = l \in \mathbb{N}$. For given values of n and l, this method requires the computation by means of recurrence relations of two auxiliary polynomials, $P(x)$ and $H(x)$, of degrees $2l - 2$ and $2l - 4$, respectively. Here it is shown that $P(x)$ is related to the coefficients of the Gaussian quadrature formula for the Gegenbauer weights $w_l(x) = (1 - x^2)^{l-1/2}$, and this fact is used to obtain the explicit expression of $P(x)$. From this result, an explicit formula is also given for the polynomial $S(x) = \lim_{n\to\infty} P\left(1 - x/(2n^2)\right)$, which is relevant to the study of the asymptotic ($n \to \infty$ with l fixed) behaviour of the entropy.

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1. Introduction

According to Shannon's information theory [1], the uncertainty associated with a continuous probability distribution with density function $\rho(\vec{x})$, $\vec{x} \in \mathbb{R}^D$, is measured by the entropy

$$S(\rho) = -\int \rho(\vec{x}) \log \rho(\vec{x})\, \mathrm{d}\vec{x}.$$

In particular, when $\rho(\vec{x})$ is the single-particle probability density for the position of a quantum system, $S(\rho)$ measures the uncertainty in the localization of the particle in position space. The momentum entropy $S(\gamma)$ can be defined likewise from the single-particle density of momentum

$\gamma(\vec{p})$. In the simplest case of a single-particle system described by the wavefunction $\psi(\vec{x})$, $\rho(\vec{x}) = |\psi(\vec{x})|^2$ and $\gamma(\vec{p}) = |\phi(\vec{p})|^2$, where the wavefunction in momentum space $\phi(\vec{p})$ is essentially the Fourier transform of $\psi(\vec{x})$. The sharp inequality

$$S(\rho) + S(\gamma) \geqslant D(1 + \log \pi)$$

provides a quantitative formulation of the position–momentum uncertainty principle which is stronger than the standard Heisenberg inequality [2].

For many important quantum systems, such as the D-dimensional harmonic oscillator and the hydrogen atom, the calculation of position and momentum information entropies involves the evaluation of integrals of the form

$$E(p_n) = -\int_a^b [p_n(x)]^2 \log[p_n(x)]^2 w(x)\,\mathrm{d}x \tag{1}$$

where $\{p_n(x)\}$ denotes a polynomial sequence ($\deg p_n(x) = n$) orthogonal on $[a, b] \subseteq \mathbb{R}$ with respect to the weight function $w(x)$. During the last decade there has been an intense activity in the study of these integrals, motivated not only by their relevance to quantum physics but also by their close relationship to other interesting mathematical objects, such as the L^p-norms or the logarithmic potentials of the polynomials $p_n(x)$. An updated survey of knowledge in this field can be found in [3].

The information entropies of the Gegenbauer or ultraspherical polynomials,

$$E\big(C_n^{(\lambda)}\big) = -\int_{-1}^1 \big[C_n^{(\lambda)}(x)\big]^2 \log\big[C_n^{(\lambda)}(x)\big]^2 (1-x^2)^{\lambda-1/2}\,\mathrm{d}x \tag{2}$$

with λ non-negative integer or half-integer, are especially relevant since they appear in the calculation of information entropies in both position and momentum spaces for any quantum-mechanical system with a central potential $V(r)$ in D-dimensional space, $D \geqslant 2$ [3–6]. We recall that the Gegenbauer polynomials $C_n^{(\lambda)}(x)$ are defined as (see, e.g., [7, section 4.7])

$$C_n^{(\lambda)}(x) = \frac{(2\lambda)_n}{\big(\lambda+\frac{1}{2}\big)_n} P_n^{(\lambda-1/2,\lambda-1/2)}(x)$$

where $P_n^{(\alpha,\beta)}(x)$ are Jacobi polynomials,

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!}\, {}_2F_1\left(\left.\begin{matrix} -n, n+\alpha+\beta+1 \\ \alpha+1 \end{matrix}\right| \frac{1-x}{2}\right)$$

and for $\lambda > -1/2$ they form an orthogonal sequence on the interval $[-1, 1]$ with respect to the weight function $w_\lambda(x) = (1-x^2)^{\lambda-1/2}$,

$$\int_{-1}^1 C_n^{(\lambda)}(x) C_m^{(\lambda)}(x)(1-x^2)^{\lambda-1/2}\,\mathrm{d}x = \frac{2^{1-2\lambda}\pi\Gamma(n+2\lambda)}{(n+\lambda)n![\Gamma(\lambda)]^2}\delta_{n,m}.$$

Instead of using the standard definition of Gegenbauer polynomials, it is often more convenient to work with the polynomials

$$\hat{C}_n^{(\lambda)}(x) = \left(\frac{\lambda(n+\lambda)n!}{(2\lambda)_n}\right)^{1/2} C_n^{(\lambda)}(x) \tag{3}$$

which are orthonormal on $[-1, 1]$ with respect to the probability density

$$\hat{w}_\lambda(x) = \frac{\Gamma(\lambda+1)}{\sqrt{\pi}\Gamma(\lambda+1/2)}(1-x^2)^{\lambda-1/2}. \tag{4}$$

The corresponding entropies are thus defined as

$$E\big(\hat{C}_n^{(\lambda)}\big) = -\int_{-1}^1 \big[\hat{C}_n^{(\lambda)}(x)\big]^2 \log\big[\hat{C}_n^{(\lambda)}(x)\big]^2 \hat{w}_\lambda(x)\,\mathrm{d}x. \tag{5}$$

The simplest particular cases of Gegenbauer polynomials are the Chebyshev polynomials of the first and second kind,

$$T_n(x) = \lim_{\lambda\to 0} \frac{n!}{(2\lambda)_n} C_n^{(\lambda)}(x) \qquad U_n(x) = C_n^{(1)}(x). \tag{6}$$

For both of these families, information entropies can be easily computed in closed analytical form, the results being [4, 5]

$$\begin{aligned} E(\hat{T}_n) &= -\frac{1}{\pi}\int_{-1}^{1} [\hat{T}_n(x)]^2 \log[\hat{T}_n(x)]^2 (1-x^2)^{-1/2}\,\mathrm{d}x = -1 + \log 2 \qquad n \geqslant 1 \\ E(\hat{U}_n) &= -\frac{2}{\pi}\int_{-1}^{1} [\hat{U}_n(x)]^2 \log[\hat{U}_n(x)]^2 (1-x^2)^{1/2}\,\mathrm{d}x = -\frac{n}{n+1}. \end{aligned} \tag{7}$$

At first sight, there is little hope to find expressions of this kind for the entropies of Gegenbauer polynomials with $\lambda \neq 0, 1$. However, in the $\lambda = 2$ case, it was proved by Buyarov [8] that

$$E\big(\hat{C}_n^{(2)}\big) = -\log\left(\frac{3(n+1)}{n+3}\right) - \frac{n(n^2+2n-1)}{(n+1)(n+2)(n+3)} - \frac{2}{\sqrt{(n+1)^3(n+3)^3}} \frac{T'''_{n+2}(\xi)}{T''_{n+2}(\xi)} \tag{8}$$

where

$$\xi = \frac{n+2}{\sqrt{(n+1)(n+3)}} \tag{9}$$

and the previous result was later simplified to [9]

$$E\big(\hat{C}_n^{(2)}\big) = -\log\left(\frac{3(n+1)}{n+3}\right) - \frac{n^3-5n^2-29n-27}{(n+1)(n+2)(n+3)} - \frac{1}{n+2}\left(\frac{n+3}{n+1}\right)^{n+2}. \tag{10}$$

In the same work [9], equation (8) was generalized to arbitrary integer values of the parameter. Following the notation of [9], for fixed $n, l \in \mathbb{N}$, $l \geqslant 2$, we consider the polynomial sequence $\{P_k(x)\}$ ($\deg P_k(x) = k$) generated by the recurrence relation

$$P_{k+1}(x) = (2l-2k-3)x P_k(x) - (n+k+1)(n+2l-k-1)(1-x^2) P_{k-1}(x) \tag{11}$$

from the initial values $P_{-1}(x) = 0$, $P_0(x) = 1$. We also introduce the additional definitions

$$\begin{aligned} P(x) &\equiv P_{2l-2}(x) = \alpha_{nl} \prod_{j=1}^{2l-2} (x - \xi_j) \\ H(x) &\equiv \sum_{k=0}^{2l-2} (-1)^k P_{k-1}(x) P_{2l-k-3}(x) = \beta_{nl} x^{2l-4} + \cdots \end{aligned} \tag{12}$$

denote by g_{nl} the leading coefficient of the Gegenbauer polynomial $\hat{C}_n^{(l)}(x)$,

$$\hat{C}_n^{(l)}(x) = g_{nl} x^n + \cdots \qquad g_{nl} = \frac{2^n (n+l-1)!}{l!} \left(\frac{l(n+l)(2l-1)!}{(n+2l-1)!\,n!}\right)^{1/2} \tag{13}$$

and use the shorthand notation

$$u_{nl} \equiv \sum_{k=l}^{2l-1} \frac{1}{n+k} = \psi(n+2l) - \psi(n+l). \tag{14}$$

Then it can be shown [9] that the entropy of the Gegenbauer polynomial $\hat{C}_n^{(l)}(x)$ is given by

$$E\big(\hat{C}_n^{(l)}\big) = -s_{nl} - r_{nl} \sum_{j=1}^{2l-2} \big(1 - \xi_j^2\big) \frac{H(\xi_j)}{P'(\xi_j)} \frac{C_{n-1}^{(l+1)}(\xi_j)}{C_n^{(l)}(\xi_j)} \tag{15}$$

where the constants s_{nl} and r_{nl} are defined as

$$\begin{aligned} s_{nl} &= 2\log\left(\frac{g_{nl}}{2^n}\right) - \frac{n}{n+l} + 2n(n+l)\frac{\beta_{nl}}{\alpha_{nl}} + 2nu_{nl} \\ r_{nl} &= 2(n+l)\sqrt{\frac{2n(l+1)(n+2l)}{2l+1}}. \end{aligned} \tag{16}$$

The explicit expressions of the polynomials $P(x)$ and $H(x)$ for arbitrary values of n and l were not given in [9]. The aim of the present paper is to provide a partial solution to this problem: in section 2, we first show that $P(x)$ is related to the coefficients of the Gaussian quadrature formula with the Gegenbauer weights $w_l(x) = (1-x^2)^{l-1/2}$, and then we take advantage of this fact in order to obtain the explicit expression of $P(x)$. From this result, we easily find the explicit form of the leading coefficients α_{nl}. In section 3, we also obtain the explicit expression of the polynomial

$$S(x) = \lim_{n\to\infty} P\left(1 - \frac{x}{2n^2}\right) \tag{17}$$

which plays a key role in the study of the asymptotic ($n \to \infty$ with l fixed) behaviour of $E\big(C_n^{(l)}\big)$ [9]. Finally, in section 4, some concluding remarks are given and several open problems suggested by the present research are pointed out.

2. Gaussian quadrature with Gegenbauer weights and the polynomial $P(x)$

For fixed $n, l \in \mathbb{N}$ ($n \geqslant 1, l \geqslant 2$), let us denote by x_j ($j = 1, \ldots, n$) the jth zero of the Gegenbauer polynomial $C_n^{(l)}(x)$. It was proved in [9] that

$$\frac{T_{n+l}^{(l-s)}(x_j)}{T_{n+l}(x_j)} = \frac{(n+l)(n+2l-s-1)!}{(n+s)!\big(1-x_j^2\big)^{(l-s)/2}}\frac{p_{s-1}(y_j)}{p_{l-1}(y_j)} \qquad 0 \leqslant s \leqslant l \tag{18}$$

$$[T_{n+l}(x_j)]^2 = (-1)^{l-1}\frac{[p_{l-1}(y_j)]^2}{p_{2l-2}(y_j)} \tag{19}$$

where the variable y and the polynomials $p_k(y)$ can be defined from x and $P_k(x)$ by means of the relations

$$y = \frac{x}{\sqrt{1-x^2}} \qquad p_k(y) = \frac{P_k(x)}{(1-x^2)^{k/2}}. \tag{20}$$

Recalling the first equation in (12), and using (20) to write (18) and (19) in terms of x and $P_k(x)$, these two equations read

$$\frac{T_{n+l}^{(l-s)}(x_j)}{T_{n+l}(x_j)} = \frac{(n+l)(n+2l-s-1)!}{(n+s)!}\frac{P_{s-1}(x_j)}{P_{l-1}(x_j)} \qquad 0 \leqslant s \leqslant l \tag{21}$$

$$[T_{n+l}(x_j)]^2 = (-1)^{l-1}\frac{[P_{l-1}(x_j)]^2}{P(x_j)}. \tag{22}$$

Combination of equations (21) and (22) yields the following generalization of the latter,

$$\big[T_{n+l}^{(l-s)}(x_j)\big]^2 = (-1)^{l-1}\left(\frac{(n+l)(n+2l-s-1)!}{(n+s)!}\right)^2\frac{[P_{s-1}(x_j)]^2}{P(x_j)} \qquad 0 \leqslant s \leqslant l \tag{23}$$

which in the particular case $s = 1$ can be written as

$$P(x_j) = (-1)^{l-1}\left(\frac{(n+l)(n+2l-2)!}{(n+1)!T_{n+l}^{(l-1)}(x_j)}\right)^2. \tag{24}$$

From the differentiation formula for Gegenbauer polynomials [7, p 81]

$$\frac{\mathrm{d}C_n^{(\lambda)}(x)}{\mathrm{d}x} = 2\lambda C_{n-1}^{(\lambda+1)}(x)$$

and the first equation of (6), we see that Gegenbauer polynomials with integer parameter are essentially derivatives of the Chebyshev polynomials of the first kind,

$$C_n^{(l)}(x) = \frac{T_{n+l}^{(l)}(x)}{2^{l-1}(n+l)(l-1)!}.$$

Equation (24) can thus be written equivalently in terms of Gegenbauer polynomials as

$$P(x_j) = (-1)^{l-1}\left(\frac{(n+2l-2)!}{2^{l-2}(l-2)!(n+1)!C_{n+1}^{(l-1)}(x_j)}\right)^2. \tag{25}$$

The Gaussian quadrature formula establishes that, if $\{p_n(x)\}$ is a polynomial sequence orthogonal on $[a, b]$ with respect to the weight function $w(x)$, then there exists a uniquely determined set of real numbers $a_1, a_2, \ldots, a_n$ (sometimes called Christoffel numbers) such that the equality

$$\int_a^b f(x)w(x)\,\mathrm{d}x = \sum_{j=1}^n a_j f(x_j)$$

holds whenever $f(x)$ is a polynomial of degree $\leqslant 2n-1$ (see, e.g., [7, section 3.4]). The nodes x_j are the zeros of $p_n(x)$, which are known to be real and simple, and the coefficients a_j are given by [7, p 48]

$$a_j = -\frac{\kappa_{n+1}}{\kappa_n}\frac{1}{p_{n+1}(x_j)p_n'(x_j)} = \frac{\kappa_n}{\kappa_{n-1}}\frac{1}{p_{n-1}(x_j)p_n'(x_j)}$$

where κ_n is the leading coefficient of $p_n(x)$, $p_n(x) = \kappa_n x^n + \cdots$.

In particular, the coefficients a_j of the Gaussian quadrature formula for the Gegenbauer weights $w_\lambda(x) = (1-x^2)^{\lambda-1/2}$ are defined uniquely by the requirement that

$$\int_{-1}^1 f(x)(1-x^2)^{\lambda-1/2}\,\mathrm{d}x = \sum_{j=1}^n a_j f(x_j) \tag{26}$$

holds if $f(x)$ is a polynomial of degree $\leqslant 2n-1$. The nodes x_j are the zeros of $C_n^{(\lambda)}(x)$, while the Christoffel coefficients are given by [7, p 352]

$$a_j = \frac{2^{2-2\lambda}\pi\Gamma(n+2\lambda)}{n![\Gamma(\lambda)]^2}\frac{1}{\left(1-x_j^2\right)\left[C_n^{(\lambda)\prime}(x_j)\right]^2}. \tag{27}$$

When $\lambda = l \in \mathbb{N}$, comparison of the previous expression with (25) reveals that there is a simple relation between $P(x_j)$ and a_j,

$$P(x_j) = \frac{(-1)^{l-1}(n+2l-1)!}{\pi n!\left(1-x_j^2\right)}a_j. \tag{28}$$

It is well known that, in the two Chebyshev cases ($\lambda = 0, 1$), the coefficients a_j have very simple analytical expressions. For $\lambda = 0$ [7, p 352]

$$a_j = \frac{\pi}{n}$$

while for $\lambda = 1$ [7, p 353]

$$a_j = \frac{\pi}{n+1}\left(1-x_j^2\right) = \frac{\pi}{n+1}\sin^2\frac{j\pi}{n+1}$$

the second expression on the right-hand side being obtained from the first one by taking into account that the zeros of $C_n^{(1)}(x) = U_n(x)$ can be exactly evaluated as $x_j = \cos[j\pi/(n+1)]$. At first sight, it seems unlikely that simple formulae of this kind can be found for other values of λ. However, in a striking parallelism with the problem of information entropies, Förster and Petras [10] proved that in the $\lambda = 2$ case the coefficients a_j also have a representation as elementary functions of the zeros x_j, and this result was later generalized to arbitrary integer values of λ.

The Petras formula [11] for the coefficients a_j establishes that, if $\lambda = l \in \mathbb{N}$, then

$$a_j = \frac{\pi}{n+l}\left(1-x_j^2\right)^l \left[1 + \sum_{\mu=1}^{l-1} \frac{1}{\left(1-x_j^2\right)^\mu}\left(\frac{(2\mu)!}{2^\mu \mu!}\right)^2 \binom{l-1+\mu}{2\mu} \prod_{k=1}^{\mu} \frac{1}{(n+l)^2-k^2}\right]. \qquad (29)$$

Using the well-known identities for the Gamma function, binomial coefficients and the Pochhammer symbol,

$$(x)_n = x(x+1)\cdots(x+n-1) = \frac{\Gamma(x+n)}{\Gamma(x)} = \frac{(-1)^n\Gamma(1-x)}{\Gamma(1-x-n)}$$

$$\binom{z}{k} = \frac{(-1)^k(-z)_k}{k!} \qquad k! = \Gamma(k+1) \qquad \Gamma(2z) = \frac{2^{2z-1}\Gamma\left(z+\frac{1}{2}\right)\Gamma(z)}{\Gamma\left(\frac{1}{2}\right)}$$

equation (29) can be written in the more compact form

$$a_j = \frac{\pi}{n+l}\sum_{\mu=0}^{l-1}\frac{(1-l)_\mu(l)_\mu(1/2)_\mu}{(1-n-l)_\mu(1+n+l)_\mu \mu!}\left(1-x_j^2\right)^{l-\mu}. \qquad (30)$$

Substitution of this equation into (28) gives

$$P(x_j) = \frac{(-1)^{l-1}(n+2l-1)!}{(n+l)n!}\sum_{\mu=0}^{l-1}\frac{(1-l)_\mu(l)_\mu(1/2)_\mu}{(1-n-l)_\mu(1+n+l)_\mu \mu!}\left(1-x_j^2\right)^{l-1-\mu}. \qquad (31)$$

Since both sides of (31) are polynomials of the same degree, we conclude that the explicit expression of the polynomial $P(x)$ is

$$P(x) = \frac{(-1)^{l-1}(n+2l-1)!}{(n+l)n!}\sum_{\mu=0}^{l-1}\frac{(1-l)_\mu(l)_\mu(1/2)_\mu}{(1-n-l)_\mu(1+n+l)_\mu \mu!}(1-x^2)^{l-1-\mu}. \qquad (32)$$

From the $\mu = 0$ term of the sum in (32), we easily find the explicit expression of the leading coefficient α_{nl},

$$\alpha_{nl} = \frac{(n+2l-1)!}{(n+l)n!} \qquad (33)$$

which is required to compute the constants s_{nl} defined in (16). The order of the terms in (32) can be reversed by writing

$$(a)_{m-r} = \frac{(-1)^r(a)_m}{(1-a-m)_r}$$

which leads to the alternative expression

$$P(x) = (-1)^{l-1}(l)_{l-1}(1/2)_{l-1}\sum_{k=0}^{l-1}\frac{(1-l)_k(1-n-2l)_k(n+1)_k}{(2-2l)_k(3/2-l)_k k!}(1-x^2)^k. \qquad (34)$$

Finally, we point out that the sums in (32) and (34) can both be written as partial sums of ${}_3F_2$ hypergeometric series. However, they cannot be written as complete hypergeometric series, since replacing the upper limit $l-1$ by infinity would give rise to additional finite terms for $\mu \geqslant n+l$ and $2l-1 \leqslant k \leqslant n+2l-1$, respectively. Using Slater's notation for the truncated hypergeometric series [12, p 83]

$$ {}_pF_q\left(\begin{matrix}\alpha_1, \alpha_2, \ldots, \alpha_p\\ \beta_1, \beta_2, \ldots, \beta_q\end{matrix}\middle| z\right)_m = \sum_{k=0}^{m} \frac{(\alpha_1)_k(\alpha_2)_k \cdots (\alpha_p)_k z^k}{(\beta_1)_k(\beta_2)_k \cdots (\beta_q)_k k!} $$

equations (32) and (34) read

$$ \begin{aligned} P(x) &= \frac{(-1)^{l-1}(n+2l-1)!}{(n+l)n!}(1-x^2)^{l-1}\, {}_3F_2\left(\begin{matrix}1-l, l, 1/2\\ 1-n-l, 1+n+l\end{matrix}\middle| \frac{1}{1-x^2}\right)_{l-1} \\ &= (-1)^{l-1}(l)_{l-1}(1/2)_{l-1}\, {}_3F_2\left(\begin{matrix}1-l, 1-n-2l, n+1\\ 2-2l, 3/2-l\end{matrix}\middle| 1-x^2\right)_{l-1}. \end{aligned} \tag{35} $$

3. Asymptotics of the entropy and the polynomial $S(x)$

Following again the notation of [9], for fixed $l \in \mathbb{N},\ l \geqslant 2$, we consider the polynomial sequence $\{S_k(x)\}$ ($\deg S_k(x) =$ integer part of $k/2$) generated by the recurrence relation

$$ S_{k+1}(x) = (2l-2k-3)S_k(x) - xS_{k-1}(x) \tag{36} $$

from the initial values $S_{-1}(x) = 0,\ S_0(x) = 1$. From the analogy of (12), we also define the polynomials

$$ S(x) \equiv S_{2l-2}(x) \qquad R(x) \equiv \sum_{k=0}^{2l-2}(-1)^k S_{k-1}(x)S_{2l-k-3}(x). \tag{37} $$

It was shown in [9] that, as $n \to \infty$,

$$ E\big(\hat{C}_n^{(l)}\big) = 1 + \log\left(\frac{(2l-1)!}{(l-1)!l!}\right) + \frac{\gamma_l}{n} + \mathrm{O}(n^{-2}) \tag{38} $$

where

$$ \gamma_l = -2l^2 + l - 2\sum_{j=1}^{l-1} \sqrt{\zeta_j}\, \frac{R(\zeta_j)}{S'(\zeta_j)} \frac{J_{l+1/2}(\sqrt{\zeta_j})}{J_{l-1/2}(\sqrt{\zeta_j})} \tag{39} $$

ζ_j being the zeros of $S(x)$ and $J_\nu(x)$ the Bessel function of order ν. Equation (38) substantially improves on previous asymptotic results for $E\big(\hat{C}_n^{(\lambda)}\big)$ with arbitrary $\lambda \in \mathbb{R}$, which only provided the leading term of the expansion [4, 13]. In turn, a refinement of (38) that also gives the explicit form of the next ($\mathrm{O}(n^{-2})$) term in the expansion as a function of the polynomials $S(x)$ and $R(x)$ has been found very recently [14].

The explicit expressions of $S(x)$ and $R(x)$ for arbitrary values of l were not given in [9]. However, it was shown therein that the polynomial $S_j(x)$ can be obtained from $P_j(x)$ by means of the limit relation

$$ S_j(x) = \lim_{n\to\infty} P_j\left(1 - \frac{x}{2n^2}\right) \tag{40} $$

locally uniform on $\mathbb{C}$, which includes (17) as the particular case $j = 2l-2$. This equation can be used to find from (32) the explicit expression of $S(x)$. To achieve this goal, we first note that

$$ x = 1 - \frac{z}{2n^2} \quad\Longrightarrow\quad (1-x^2)^{l-1-\mu} = \left(\frac{z}{n^2}\right)^{l-1-\mu}\left(1 - \frac{z}{4n^2}\right)^{l-1-\mu}. $$

On the other hand, from (32) we readily see that the coefficient of $(1-x^2)^{l-1-\mu}$ in $P(x)$ is a polynomial of degree $2(l-1-\mu)$ in n, since

$$\frac{(n+2l-1)!}{(n+l)n!}\frac{1}{(1-n-l)_\mu(1+n+l)_\mu} = (-1)^\mu n^{2(l-1-\mu)} + \cdots.$$

A straightforward calculation using (17) and (32) then leads to the following expression:

$$S(x) = \sum_{\mu=0}^{l-1} \frac{(1-l)_\mu (l)_\mu (1/2)_\mu}{\mu!}(-x)^{l-1-\mu} \tag{41}$$

which is essentially a terminating ${}_3F_0$ hypergeometric series,

$$S(x) = (-x)^{l-1}{}_3F_0\left(\left.\begin{matrix}1-l, l, 1/2\\ -\end{matrix}\right| -\frac{1}{x}\right). \tag{42}$$

Finally, reversing the series in (41) we obtain the alternative expression

$$S(x) = (-1)^{l-1}(l)_{l-1}(1/2)_{l-1}\sum_{k=0}^{l-1}\frac{(1-l)_k}{(2-2l)_k(3/2-l)_k k!}(-x)^k. \tag{43}$$

This sum is not a complete hypergeometric series, since replacing the upper limit $l-1$ by infinity would give rise to additional finite terms for $k \geqslant 2l-1$. It can be written as a partial sum of a ${}_1F_2$ hypergeometric series,

$$S(x) = (-1)^{l-1}(l)_{l-1}(1/2)_{l-1}{}_1F_2\left(\left.\begin{matrix}1-l\\ 2-2l, 3/2-l\end{matrix}\right| -x\right)_{l-1}. \tag{44}$$

4. Concluding remarks and open problems

The main result obtained in this paper is equation (28), which relates the polynomial $P(x) = P_{2l-2}(x)$ generated by (11) to the Christoffel coefficients of the Gaussian quadrature formula for the Gegenbauer weight $(1-x^2)^{l-1/2}$. Taking advantage of the Petras formula (29), we then were able to find the explicit expression of $P(x)$. In turn, use of this result together with the limit relation (17) allowed us to obtain the explicit expression of the polynomial $S(x) = S_{2l-2}(x)$ generated by (36).

The fact that such expressions do exist strongly suggests that the same should happen for every polynomial in the sequences $\{P_k(x)\}$ and $\{S_k(x)\}$ generated by the three-term recurrence relations (11) and (36), respectively. However, up to now we have not been able to find the general expressions of $P_k(x)$ and $S_k(x)$ for $1 \leqslant k \leqslant 2l-2$. As a consequence, we do not know the explicit expression of the polynomial $H(x)$ (resp. $R(x)$) defined by the second equation in (12) (resp. (37)), which is required to compute the exact (resp. asymptotic) value of $E\big(C_n^{(l)}\big)$ by means of equations (15) and (16) (resp. (38) and (39)).

For Gegenbauer polynomials with non-integer parameter, equation (29) provides an asymptotic approximation of arbitrary precision for the Christoffel coefficients a_j. More precisely, if $\lambda \notin \mathbb{N}$, $\lambda > -1/2$, and m is an arbitrary integer, then [11]

$$a_j = \frac{\pi}{n+\lambda}\big(1-x_j^2\big)^\lambda \left[1 + \sum_{\mu=1}^{m-1}\frac{1}{\big(1-x_j^2\big)^\mu}\left(\frac{(2\mu)!}{2^\mu \mu!}\right)^2\binom{\lambda-1+\mu}{2\mu}\prod_{k=1}^{\mu}\frac{1}{(n+\lambda)^2-k^2} \right.$$
$$\left. + \mathrm{O}\big(n^{-2m}\big(1-x_j^2\big)^{-m}\big)\right].$$

One may wonder whether this result can be useful to study the asymptotics of the information entropies $E\left(C_n^{(\lambda)}\right)$ for arbitrary $\lambda \in \mathbb{R}$. Finally, another open problem is that of extending the relationship between information entropies and Gaussian quadrature formulae to other families of orthogonal polynomials.

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References

[1] Shannon C E 1948 A mathematical theory of communication *Bell Syst. Tech. J.* **27** 379–423, 623–56
Reprinted in Shannon C E and Weaver W (ed) 1949 *The Mathematical Theory of Communication* (Urbana, IL: University of Illinois Press)
[2] Beckner W 1975 Inequalities in Fourier analysis *Ann. Math.* **102** 159–82
Bialynicki-Birula I and Mycielski J 1975 Uncertainty relations for information entropy in wave mechanics *Commun. Math. Phys.* **44** 129–32
[3] Dehesa J S, Martínez-Finkelshtein A and Sánchez-Ruiz J 2001 Quantum information entropies and orthogonal polynomials *J. Comput. Appl. Math.* **133** 23–46
[4] Yáñez R J, Van Assche W and Dehesa J S 1994 Position and momentum information entropies of the *D*-dimensional harmonic oscillator and hydrogen atom *Phys. Rev.* A **50** 3065–79
[5] Dehesa J S, Van Assche W and Yáñez R J 1997 Information entropy of classical orthogonal polynomials and their application to the harmonic oscillator and Coulomb potentials *Meth. Appl. Anal.* **4** 91–110
[6] Yáñez R J, Van Assche W, González-Férez R and Dehesa J S 1999 Entropic integrals of hyperspherical harmonics and spatial entropy of *D*-dimensional central potentials *J. Math. Phys.* **40** 5675–86
[7] Szegö G 1975 *Orthogonal Polynomials (Am. Math. Soc. Colloq. Publ. vol 23)* 4th edn (Providence, RI: American Mathematical Society)
[8] Buyarov V S 1997 On information entropy of Gegenbauer polynomials *Vestn. Mosk. Univ. Ser. I Mat. Mekh.* **6** 8–11 (in Russian)
[9] Buyarov V S, López-Artés P, Martínez-Finkelshtein A and Van Assche W 2000 Information entropy of Gegenbauer polynomials *J. Phys. A: Math. Gen.* **33** 6549–60
[10] Förster K-J and Petras K 1990 On estimates for the weights in Gaussian quadrature in the ultraspherical case *Math. Comput.* **55** 243–64
[11] Petras K 1996 An asymptotic expansion for the weights of Gaussian quadrature formulae *Acta Math. Hung.* **70** 89–100
[12] Slater L J 1966 *Generalized Hypergeometric Functions* (Cambridge: Cambridge University Press)
[13] Aptekarev A I, Dehesa J S and Yáñez R J 1994 Spatial entropy of central potentials and strong asymptotics of orthogonal polynomials *J. Math. Phys.* **35** 4423–8
Aptekarev A I, Buyarov V S and Dehesa J S 1995 Asymptotic behaviour of L^p-norms and entropy for general orthogonal polynomials *Russ. Acad. Sci. Sb. Math.* **82** 373–95
Aptekarev A I, Buyarov V S, Van Assche W and Dehesa J S 1996 Asymptotics of entropy integrals for orthogonal polynomials *Dokl. Math.* **53** 47–9
[14] Sánchez-Lara J F 2002 On the asymptotic expansion of the entropy of Gegenbauer polynomials *J. Comput. Appl. Math.* **142** 401–9